# ML-DOA estimation using a sparse representation of array covariance

Thomas Aussaguès[1,2], Anne Ferréol[1,2], Alice Delmer[1], and Pascal Larzabal[2]

[1] Thales, 4 avenue des Louvresses, 92230 Gennevilliers, France
[2] SATIE, Université Paris-Saclay, UMR CNRS 8029, 4 avenue des Sciences, 91190 Gif-sur-Yvette, France
thomas.aussagues@ens-paris-saclay.fr

**Abstract.** Sparse Direction-of-Arrival (DOA) estimators depend on the regularization parameter $\lambda$ which is often empirically tuned. In this work, conducted under the vectorized covariance matrix model, we are looking for theoretical equivalence between the Maximum Likelihood (ML) and sparse estimators. We show that under mild conditions, $\lambda$ can be chosen thanks to the distribution of the minimum of the ML criterion. The corresponding $\lambda$ choice is $\theta$-invariant, only requiring an upper bound on the number of sources. Furthermore, it guarantees the global minimum of the sparse $\ell_0$-regularized criterion to be the ML solution.

**Keywords:** Sparse estimation · Regularization parameter · Maximum Likelihood.

## 1 Motivation

DOA estimation is a pivotal area in numerous critical applications such as radar or telecommunications. Throughout the last 60 years, a plethora of estimation techniques has been proposed [3]. Nevertheless, classical methods such as MUSIC [1] have limited performances in severe conditions (few array snapshots, coherent sources, low Signal-to-Noise Ratio (SNR) or modeling errors). Although it achieves the Cramér-Rao Lower Bound at SNR, the ML estimator [4] is rarely employed as it requires multi-dimensional highly non-convex optimization.

In recent years, there has been a growing interest within the signal processing processing community on sparse methods applied to DOA estimation [6] as they exhibit enhanced performances in tough scenarios. Out of the numerous modeling of the sparse DOA estimation problem, the sparse covariance matrix representation [5] emerged as a promising choice.

For this model, the DOAs can be estimated through the minimization of a non-convex $\ell_0$-regularized objective $\mathcal{J}_{\ell_0}$ which is parametrized by $\lambda$ the regularization parameter. Many works empirically tuned $\lambda$ which is unfeasible in practice. In [2], the authors proposed an interval for the regularization parameter. However, the interval bounds depends on the sources directions thus complexifying off-line selection of the regularization parameter.

In this work, we introduce a novel $\theta$-invariant regularization parameter choice $\lambda$ relying on the ML criterion distribution.

## 2 Sparse modeling & sparse estimation

Let us consider $M$ narrowband sources of directions $\boldsymbol{\Theta} = \{\theta_1, \ldots, \theta_M\}$ impinging on an array of $N$ antennas and $\boldsymbol{\Phi} = \{\varphi_1, \ldots, \varphi_G\}$ a grid of $G$ pre-defined directions such that $\boldsymbol{\Theta} \subset \boldsymbol{\Phi}$. According to [5], the sparse covariance matrix model is then:

$$\mathbf{r} = \mathbf{B}(\boldsymbol{\Phi})\boldsymbol{\gamma}_0 + \boldsymbol{\delta} \tag{1}$$

where $\mathbf{y} \in \mathbb{C}^{N^2 \times 1}$ is the observation, $\mathbf{B}(\boldsymbol{\Phi}) \in \mathbb{C}^{N^2 \times G}$ a dictionary containing the array response for each grid direction, $\gamma_0 \in \mathbb{C}^{G\times}$ $M$-sparse vector and $\boldsymbol{\delta_w}$ a complex non-white Gaussian noise. We propose to apply a whitening transform, defined by $\widehat{\mathbf{W}}$, to (1). This transform enables equivalence between ML and (2) as detailed in section 3.

$$\mathbf{y} = \mathbf{W}\mathbf{r} = \widehat{\mathbf{W}}\mathbf{B}(\boldsymbol{\Phi})\boldsymbol{\gamma}_0 + \boldsymbol{\delta_w} \tag{2}$$

with $\boldsymbol{\delta_w} = \widehat{\mathbf{W}}\boldsymbol{\delta}$ a white complex Gaussian noise vector.

Finally, the DOAs are estimated from the indices of the non-zero components of $\boldsymbol{\gamma}_0$ which correspond to sources directions. Following Delmer's work, $\boldsymbol{\gamma}_0$ is estimated through the minimization of the following $\ell_0$-penalized criterion:

$$\min_{\boldsymbol{\gamma} \in \mathbb{C}^G} \left\{ \mathcal{J}_{\ell_0}(\lambda, \boldsymbol{\gamma}) = \frac{1}{2}\|\mathbf{y} - \widehat{\mathbf{W}}\mathbf{B}(\boldsymbol{\Phi})\boldsymbol{\gamma}\|_2^2 + \lambda\|\boldsymbol{\gamma}\|_0 \right\} \tag{3}$$

with a suitable regularization parameter $\lambda > 0$.

## 3 A novel $\boldsymbol{\theta}$-invariant regularization parameter $\boldsymbol{\lambda}$ choice

The choice of $\lambda$ in (3) is of utmost importance since it controls the trade-off between data fidelity and sparsity. For the $\ell_0$ optimization framework„ $\lambda$ is generally empirically tuned given that only few results are available. Recently, Delmer [2] *et al.* proposed an interval $[\lambda^-, \lambda^+]$ for the regularization parameter. For the case of $M = 2$ impinging sources, we show that, after the whitening transform, choosing $\lambda$ within the aforementioned interval ensures that $\mathcal{J}_{\ell_0}$ global minimizer corresponds to the ML solution. Both criteria are thus said equivalent.

Nevertheless, the statistics of the interval bounds $\lambda^-, \lambda^+$ are difficult to compute as they require the knowledge of the DOAs. To this end, we propose a novel regularization parameter choice invariant w.r.t DOAs. We show that, under mild conditions, the minimum of the ML criterion, denoted $\epsilon$ belongs to $[\lambda^-, \lambda^+]$ and thus can be use as regularization parameter.

Specifically, we prove that $\epsilon$ is $\chi^2$-distributed with $N^2 - M$ degrees of freedom and that $\epsilon$ distribution does not depend on the directions. Consequently, we propose to choose $\lambda$ using $\epsilon$ distribution:

$$\lambda = \frac{1}{2}F_\epsilon(\eta) \tag{4}$$

where $F_\epsilon$ is the inverse Cumulative Distribution Function of $\epsilon$ for probability $\eta$.

Numerically, we verify on figure Fig. 1 that, with the proposed $\lambda$ choice and $\eta = 0.05$, the sparse estimator yields the same statistical performance as the ML estimator for SNR $\geq -4\,$dB. Additional simulations demonstrate that both estimators yield the same estimates hence confirming the equivalence.

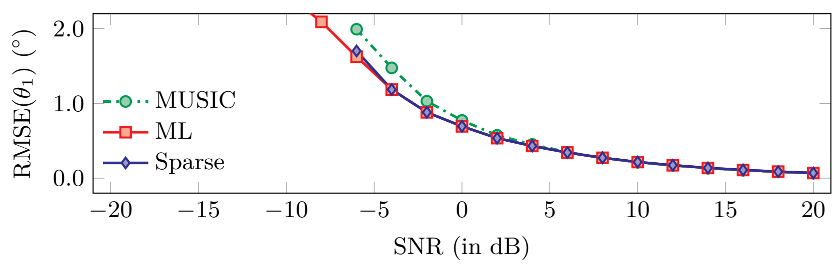


**Fig. 1.** Root Mean Square Error (RMSE) for direction $\theta_1$. An array of $N = 6$ antennas with 5 antennas distributed around a circle of radius $0.8\lambda_0$ where $\lambda_0$ denotes the wavelength and one central sensor is considered. $M = 2$ sources of directions $\theta_1 = 180^\circ, \theta_2 = 200^\circ$ with $K = 200$ array snapshots impinge on this array.

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